Bootstrapping to Calculate Fourier Coefficients of Paramodular Forms:
A Computational Aspect of Modularity

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Motivation

The goal of this project is to use a bootstrapping method to calculate additional Fourier coefficients of particular paramodular forms. These paramodular forms have a conjectural relationship with Abelian surfaces. Such surfaces may have practical applications to *cryptography and cryptology*. This project is related to one of the most celebrated mathematical results of the previous century, the proof of *Fermat's last theorem*.

Outline

1. First we use the Fourier-Jacobi decomposition [EZ85] of Siegel modular forms
$$F(\tau, z, \tau') = \sum_{[a,b,c]} c([a, b, c])\tau^a z^b \tau'^c = \sum_{m\geq 0} \Phi_m(\tau, z)\tau'^c.$$
2. For each $m \geq 0$ the set of Jacobi forms of weight m is a finite dimensional *vector space*. We have dimension formulas for the each of these vector spaces and we use the theory of theta blocks [Sko08] to find a spanning set.
3. Others [RSS12, BK14, PY15] have naively calculated some Fourier coefficients, which is rather laborious. With these few coefficients we are able to use basic *linear algebra* to determine what the Jacobi modular forms are.
4. With the additional Fourier coefficients just calculated we can repeat the process, this is the basic idea of bootstrapping.

Dimension

Based on the dimension formulas of ordinary modular forms, which are well understood and calculable in SAGE, we have a simple dimension formula for the *vector space* of Jacobi modular forms of weight 2 and index m. Below we collect some values.

m	277	544	831	1108	1385	1662	1939	2216	2493	2770
$\dim J^c_{2,m}$	10	17	20	37	45	45	72	73	83	89

Theta Blocks

The functions $\eta(\tau)$ and $\vartheta(\tau, z)$ are building blocks we can use to make many different Jacobi forms:
$$\eta(\tau)^{-6}\prod_{i=1}^{10}\vartheta(\tau, d_i z)$$
is a Jacobi modular form of weight 2 and of index m as long as the following two conditions hold

1. $2m = \sum_{i=1}^{10} d_i^2$
2. $\frac{1}{6} + \sum_{i=1}^{10}\bar{B}_2(d_i x)$ has a positive minimum on $[0, 1]$, where $B_2 = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ and $\bar{B}(x) = B(x - [[x]])$.

Abelian Surfaces

An *Abelian surface* is a 2 dimensional torus (i.e., a donut) which is defined over $\mathbb{C}$, which can be represented as the solution set of a set of polynomials. As such it is a 4-dimensional object over the real numbers. Below is 3-dimensional representation.

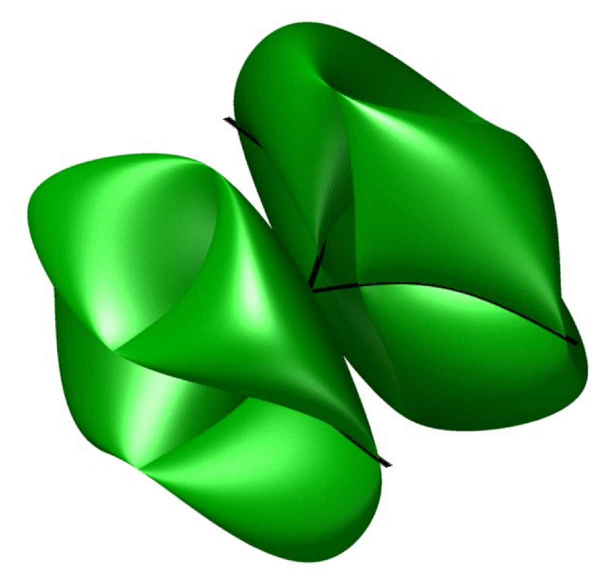

Figure: A three dimensional representation of an Abelian surface.

An *endomorphism* of an Abelian surface is a map that is locally given by polynomials. All Abelian surfaces have the endomorphisms which maps the point $P \mapsto kP$ for all $k \in \mathbb{Z}$, if an Abelian surface has more endomorphisms then this we say that it is of GL_2 type.

Paramodular Forms

A paramodular form is a complex valued function whose domain is the set of 2×2 matrices which are symmetric and whose imaginary part is *positive definite*, which has additional symmetries.

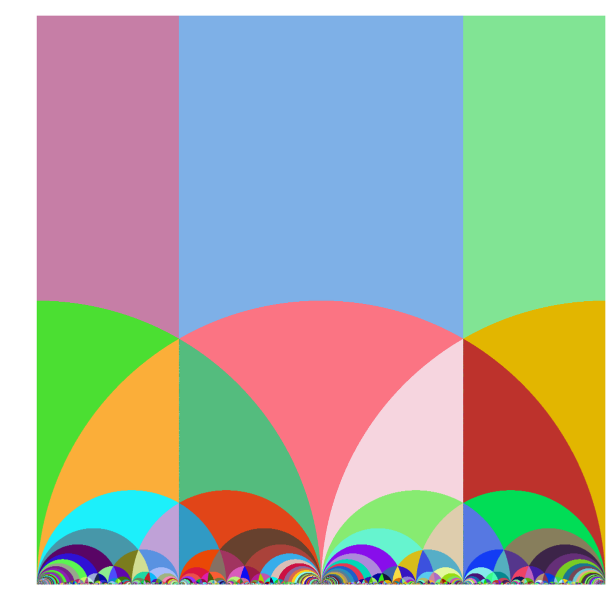

Figure: Symmetry of modular forms

Modular forms are functions on the complex plane that are inordinately symmetric. They satisfy so many internal symmetries that their mere existence seem like accidents.
- Barry Mazur

There are five fundamental operations of arithmetic, addition, subtraction, multiplication, division, and modular forms.
- Martin Eichler

Paramodular Conjecture

For every Abelian Surface defined over the rational number $\mathbb{Q}$ which is of GL_2 type has a corresponding paramodular forms. There is a large class of examples for which this conjecture has been proven [JLR12]

Numerical Analysis

Condition 1 is very easy to control for but condition 2 may be a bit harder. Let's plot some of the functions we need to verify.

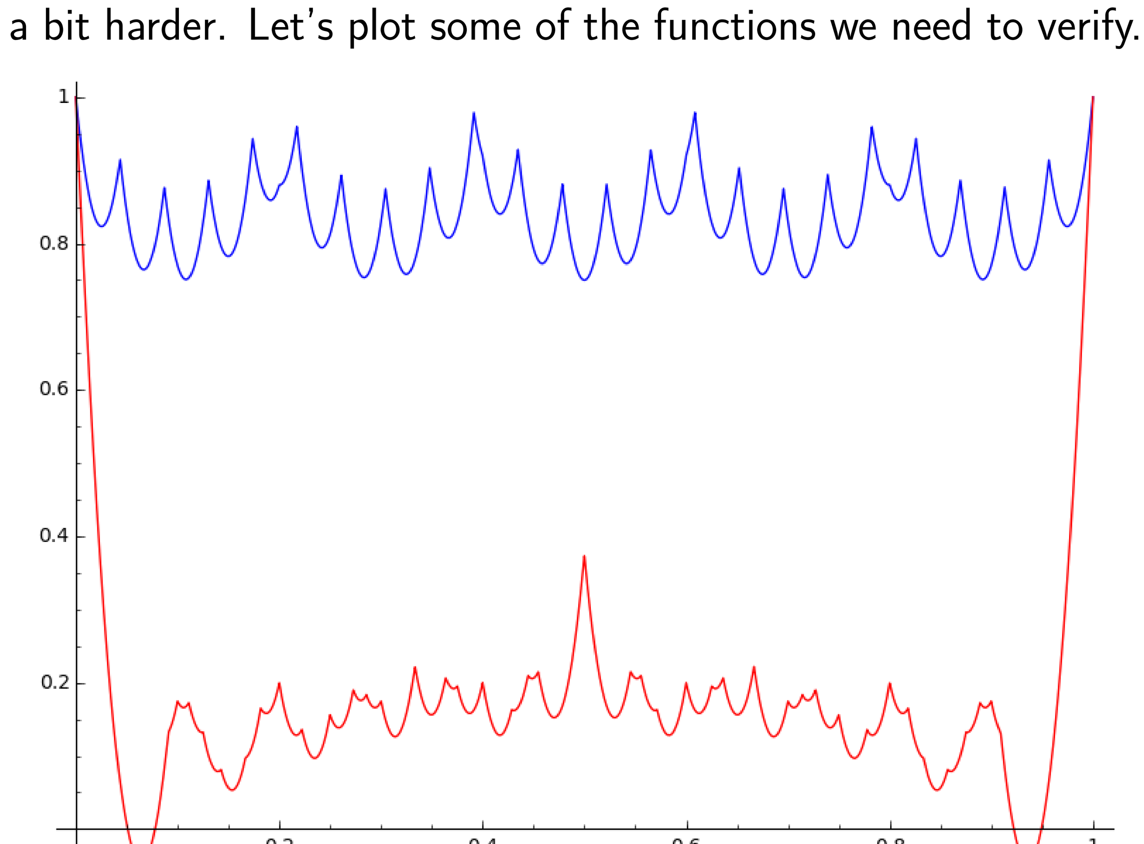

Figure: The above graph of $\frac{1}{6}[\bar{B}_2(5x) + \bar{B}_2(23x)]$ in blue indicates that we found a theta block of weight 2 and index 277. While, the graph of $\frac{1}{6} + \sum_{k=2}^{11}\bar{B}_2(kx)$ in red indicates that this does not correspond to a theta block.

An Example

From data collected by [PY15] we have reason to suspect that there is there are paramodular forms of weight 2 and index 277, and that this is the smallest index. As such there has been considerable effort into calculating its Fourier expansion and we do have enough coefficients to get our bootstrapping started. The dimensions of the space of the first 10 non-zero Jacobi cusp forms in the expansion of this paramodular form are given in a previous table.

1. We can calculate a basis of theta blocks for each of these spaces. Each theta block is approximated by a two variable polynomial of degree 100, or higher.
2. We can write the first ten, or more, Jacobi forms in this expansion as a linear combination of elements of the basis.
3. Based on the precision to which the Jacobi forms were calculated we then get thousands, if not more, Fourier coefficients.
4. This provides an order of magnitude increase in the number of previously calculated Fourier coefficients.

This same process can be applied to any paramodular form for which there are already previously calculated Fourier coefficients.

Further Work

1. Optimization of code using Cython.
2. How many Fourier coefficients are needed to boot strap?
3. Can we prove that an iterative process could theoretically determine all of the fourier coefficients.
4. Expanding and organizing the project systematically.

Additional Information

Popular references to the beautiful story of Andrew Wiles and the proof of Fermat's Last Theorem:

1. "Fermat's Last Theorem" is a BBC Horizons documentary.
2. "Fermat's Enigma" is a book by Sigon Sighn.
3. "The Proof" is a PBS documentary.

References

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Building Blocks

$$\eta(\tau) = e^{\frac{\pi i\tau}{12}}\prod_{n=1}^{\infty}(1 - q^n)$$
$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n z}$$