MALCOLM RUPERT
Research Statement

1. Introduction

I study representations of algebraic groups over local and global number fields and the connections between them in order to make inferences about modular forms. My research sits at the intersection of number theory, p-adic analysis, and representation theory. One of the motivations for this line of inquiry is to better understand the modularity of elliptic curves and higher dimensional abelian varieties. For example the modularity of elliptic curves with complex multiplication can be seen by constructing a certain representation of $\mathrm{GL}(2, \mathbb{A}_{\mathbb{Q}})$ from two Hecke characters. This is analogous, in some sense, to the construction of certain Siegel paramodular forms that appear in my thesis. In addition to the representation theoretic results in my thesis I have also worked on a project to calculate Siegel paramoduar forms using the structure of the Fourier-Jacobi decomposition.

One popular construction of Siegel paramodular forms, which is complementary to the theta lift method in my thesis, is the Gritsenko lift. This lift produces a Siegel paramodular form from a classical modular form of half-integral weight. These half integral weight modular forms are well understood and there are many that are explicitly calculated. Even so, almost all Siegel paramodular forms are not Gritsenko lifts. For these non-lifts there is no known effective way to calculate Fourier coefficients, but I want to determine to what extent my theta lift method will ameliorate the situation. To preform these calculations the data tabulated on the L-functions and Modular Forms Database will be an asset.

The L-functions and Modular Forms Database (LMFDB, [LMF13]) is an open source database that has been called the 'periodic tables of number theory' and it collects objects in four main categories: L-functions, modular forms, motives (e.g., varieties), and Galois representations. Modern modularity results, starting with the Taniyama-Shimura conjecture ([ST61]) and its proof ([Wil95, TW95, CDT99, BCDT01]), unite all four of these categories in order to assign every geometric object under consideration a modular form whose L-functions coincides with its own. The Taniyama-Shimura conjecture considered rational elliptic curves with integral coefficients and assigned a classical modular form to it. More recently this has been extended to a modularity result for elliptic curves over real quadratic number fields by [FLHS15]. The existence of the theta lift that is presented in my thesis can be used to prove the modularity result for abelian surfaces which are the Weil restriction of scalars of elliptic curves over real quadratic number fields.

2. Thesis Results

In general, a theta lift is a tool for relating representations of certain pairs of of subgroups (S_1, S_2) of a symplectic group $\mathrm{Sp}(X)$ defined over a number field. The global theta lift from S_1 to S_2 takes as input a cups form f on the adeles of S_1 and defines an automorphic form on the adeles of S_2 by integrating f against a theta kernel; this theta kernel depends on a choice of a certain Schwartz function φ. In a similar manner the theta lift can produce an automorphic form on S_1 from a cusp form on S_2. Determining a choice for φ which gives automorphic forms with desirable qualities requires a theory of local theta lifts that are commensurable with the global theory. While the global theta lift has a natural construction, as an integral operator, there is nothing so ubiquitous in the local theory. In [Wal80], Waldspurger studied the theta lift when W is a 2-dimensional symplectic space, V is a rank 3 quadratic space, and $X = W \otimes V$. He investigated the correspondence between modular forms of half integral weight and those with integral weight and, in [Wal81], computed special values of L-functions in terms of Fourier coefficients of half integral weight modular forms. My research considers the case when W is a 4-dimensional symplectic space, V is a rank 4 quadratic space, and $X = W \otimes V$. This theory gives a correspondence between Hilbert modular forms and Siegel paramodular forms. In particular, I have studied an integral formula for the local theta lifts which allows for explicit construction of the Siegel paramodular forms appearing in this correspondence.

Let $\mathcal{H}$ be the space of elements of $\mathrm{M}(2, \mathbb{C})$ whose imaginary part is positive definite and the paramodular group $\mathrm{K}(N)$ is a subgroup of $\mathrm{Sp}(\mathbb{Q}, 4)$ defined by certain congruence relations. A Siegel paramodular form is an analytic function $F : \mathcal{H} \to \mathbb{C}$ such that $F(\gamma\langle Z\rangle) = j(\gamma, Z)^{-k}F(Z)$ for all $\gamma \in \mathrm{K}(N)$ and $Z \in \mathcal{H}$, where $\gamma\langle\cdot\rangle$ is the action by fractional linear transformation. It was conjectured by Brumer and Kramer, in [BK14], that an abelian surface A with $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$ has an associated Siegel paramodular form. Poor and Yuen ([PY15]), and later Berger et al. ([BDPŞ15]), provided additional evidence for this conjecture by finding some examples of rational abelian surfaces by looking for them precisely where the paramodular conjecture predicts. The paramodular conjecture is a precise and falsifiable generalization of the Taniyama-Shiruma conjecture, to degree 2.

Theorem. *Let L be a local field and let E be a real quadratic extension. Assume that if the residual characteristic of L is even that E/L is unramified. Let f be a local newform of* $\mathrm{GL}(2, E)$*. I have an explicit choice of Schwartz function φ, depending on the splitting behavior of $\mathfrak{p}_L$ in $\mathfrak{o}_E$, which produces a non-zero paramodular invariant vector by way of the local theta lift of* $(\mathrm{SO}(4), \mathrm{GSp}(4))$.

3. Additional Results

In addition to the work on the theta lift I also worked on another aspect of computing Siegel modular forms, with Nathan Ryan, using a method we called bootstrapping. Let F is a Siegel modular form of weight k and level m, then we can write the Fourier expansion of F as

$$F(\tau, z, \tau') = \sum_{[a,b,c]} k([a, b, c])\tau^a z^b \tau'^c$$

where the index runs over positive semi-definite quadratic forms $[a, b, c]$. Furthermore, there exists Jacobi modular forms Φ_m of level m and index k for each $m \geq 0$ such that

$$F(\tau, z, \tau') = \sum_{m\geq 0} \Phi_m(\tau, z)\tau'^c.$$

The idea of bootstrapping is to use the structure of this Fourier-Jacobi expansion of F and any other invariance of F to determine the Fourier coefficients $k([a, b, c])$ of F if some are already computed. This is a practical approach because computing coefficients from scratch is prohibitively time consuming. If one can determine a Jacobi form in the Fourier-Jacobi expansion they will have found many Fourier coefficients. In the pursuit of implementing this approach I helped to developed an algorithm which will attempt to calculate a basis of Jacobi forms for a given index and level using theta blocks. This approach seems to work very well for low levels but has diminishing returns at higher levels.

4. Future Work

My immediate plans are to implement an algorithm for the explicit theta lift to calculate Fourier coefficients of paramodular forms, on a large scale. I will also take some time to study the implications of my Siegel paramodular form construction, along several fronts. Recent results of Daniel Riess [Rie16], and the joint work of Jennifer Johnson-Leung and Brooks Roberts [JLR14] give additional uses of this data. The bootstrapping method may also be useful in this endeavor, depending on the computational difficulty of the explicit theta lift.

In the future, I would like to investigate other interesting theta lifts. Berger et al. have proved the existence of a theta lift, in [BDPŞ15], which produces a Siegel paramodular form from a Bianchi modular form. Because of the similarities between this lift and the lift in my thesis, I believe that I can also make this lift explicit. Furthermore, the validity of both of these lifts relies on a global proof of the local theta correspondence. I would like to find a purely local proof, As Waldspurger did for the $(\mathrm{GSp}(2), \mathrm{SO}(3))$ theta lift. Further study of test vectors may also provide insights into the group structure of the paramodular group.

Because of the multiple computational and theoretical facets of my research I have a great variety of research projects for students. For instance, one aspect of generating Jacobi modular forms using theta blocks requires a challenging optimization problem of certain continuous real functions on $[0, 1]$. Calculating cosets and double cosets of algebraic groups is an integral part of many constructions in my research. I have the experience in and passion for research and teaching to build a program that collaborations with faculty and students. I look forward to the opportunity to imbue the next generation with an drive to understand this beautiful and curious subject.

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