Towards an Explicit Theta Lift from Hilbert to Siegel Paramodular Forms

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University of Idaho
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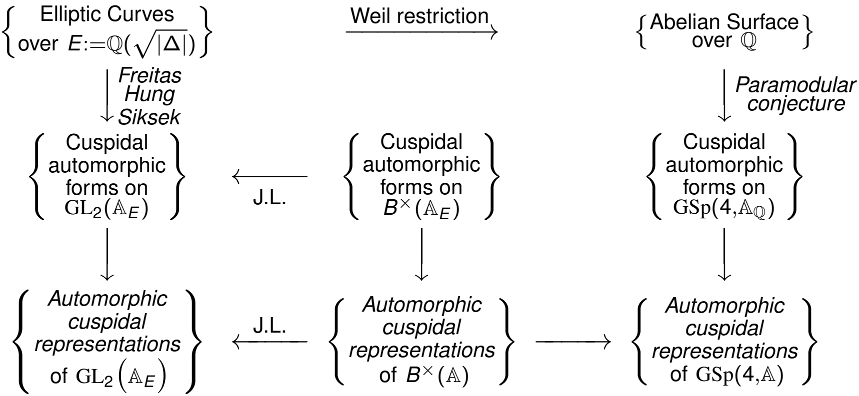
The Big Diagram



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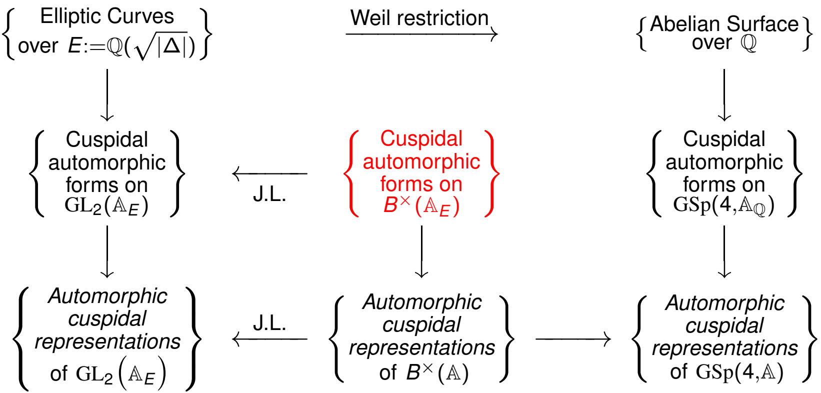

Quaternion Algebras

- Let $D = \left(\frac{\alpha,\beta}{\mathbb{Q}}\right)$ be the 4-dimensional $\mathbb{Q}$-algebra generated by $\{1, i, j, k\}$ where $i^2 = \alpha,\ j^2 = \beta,\ ij = -ji = k$.
- Localizing at a place v of $\mathbb{Q}$ gives us $D_v = \mathbb{Q}_v \otimes_{\mathbb{Q}} D$ which is either the unique 4-dimensional division algebra over $\mathbb{Q}_v$ or is the matrix algebra $\mathrm{M}_2(\mathbb{Q}_v)$.

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- Let $E = \mathbb{Q}(\sqrt{\delta})$ be a real quadratic number field with ring of integers $\mathfrak{O}$ and define $B = D \otimes_{\mathbb{Q}} E = \left(\frac{\alpha,\beta}{E}\right)$ be the extension of scalars and similarly define B_v and $\mathrm{disc}(B)$.

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- Fix a maximal order R of B and $\mathfrak{c} \lhd \mathfrak{O}$ so that $(\mathfrak{c}, \mathrm{disc}(B)) = 1$.
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$$U_v := \begin{cases} \mathrm{GL}(2, \mathfrak{o}_v) & \text{if } v \nmid \mathfrak{c} \cdot \mathrm{disc}(B) \\ \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}(2, \mathfrak{o}_v) : c \in \mathfrak{p}_v^{\mathrm{val}_v(\mathfrak{c})} \right\} & \text{if } v \mid \mathfrak{c} \\ R_v^\times & \text{if } v \mid \mathrm{disc}(B). \end{cases}$$

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Quaternionic Modular Forms

Definition

We let the space of *quaternionic automorphic forms*, $\mathcal{A}(B^{\times}, \mathfrak{c})$, be the complex vector space of functions $f : B^{\times}(\mathbb{A}_E) \to \mathbb{C}$ satisfying the following conditions:

1. $f(b_0 b) = f(b)$ for all $b_0 \in B^{\times}(E)$ and $b \in B^{\times}(\mathbb{A}_E)$;
2. $f(br) = f(b)$ for all $r \in U$ and $b \in B^{\times}(\mathbb{A}_E)$;
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More simply, when $\mathrm{disc}(B) = (1)$ one may define quaternion modular forms to be the space of functions from the class group of $R_{0,\mathfrak{c}}$ to $\mathbb{C}$, where $R_{0,\mathfrak{c}}$ is an Eichler order of level $\mathfrak{c}$.

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Quaternionic Modular Forms: Class number 1

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- Additionally $R_{\mathfrak{q}}^\times / R_{\mathfrak{c},q}^\times \simeq P^1(\mathfrak{O}_q/\mathfrak{q}^{e_\mathfrak{q}})$, given by $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mapsto (a, c)$ where $P^1(A) = \left\{(a, b) \in A^2 \mid \alpha a + \beta b = 1 \text{ for some } (\alpha, \beta) \in A^2\right\} / A^\times$.

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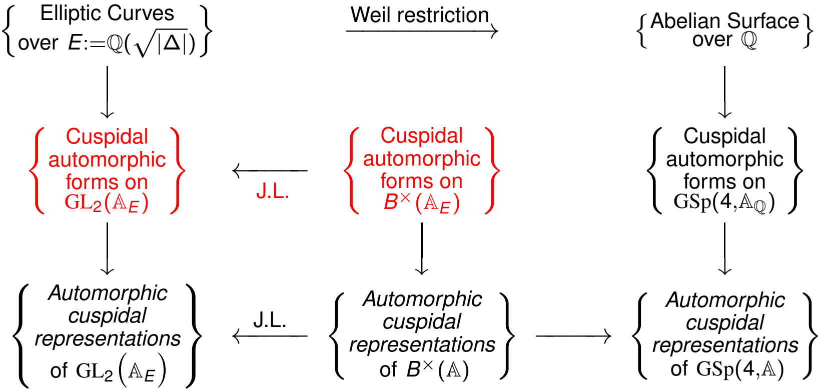

The Jaquet Langlands Correspondence

Let $\mathcal{A}_{2,2}(\mathrm{GL}_2(E), \mathfrak{c})$ denote the set of cuspidal automorphic forms over $\mathrm{GL}_2(E)$ of parallel weight two and level $\mathfrak{c}$.

Theorem (Eichler-Shimuzu-Jacquet-Langlands)

Let B be a quaternion algebra over E and let $\mathfrak{c}$ be coprime to $\mathrm{disc}(B)$. Then there is an injective map of Hecke modules

$$\mathcal{A}(B^\times, \mathfrak{c}) \hookrightarrow \mathcal{A}_{2,2}(\mathrm{GL}_2(E), \mathfrak{c})$$

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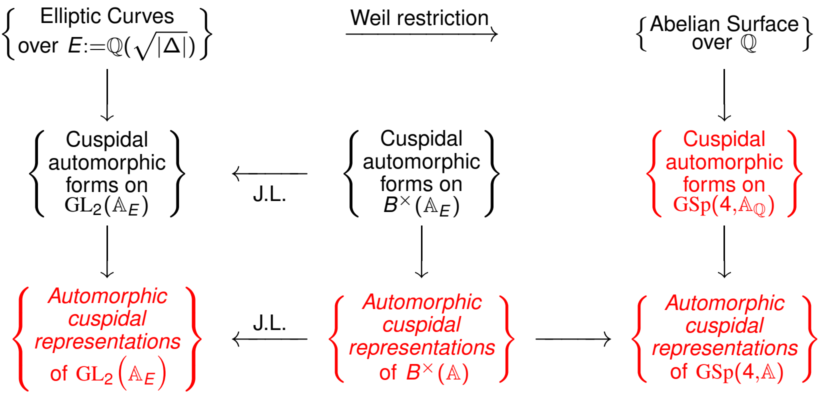

Hilbert–paramodular correspondence

Theorem (Johnson-Leung & Roberts, 2012)

Let $E/\mathbb{Q}$ be a real quadratic extension and π_0 a cuspidal, irreducible automorphic representation of $\mathrm{GL}_2(\mathbb{A}_E)$ with trivial central character and infinity type $(2, 2n+2)$ for some $n \in \mathbb{N}$. Then there exists a non-zero paramodular newform with degree 2, weight $2+n$, and level, Hecke eigenvalues, epsilon factor and L-function determined explicitly by π_0.

The Symmetric Bilinear Space

- Let $E/\mathbb{Q}$ be a quadratic field extension and set $B = E \otimes_{\mathbb{Q}} D$ where D is a quaternion algebra over $\mathbb{Q}$.
- Let $* : B \to B$ be the natural involution defined by $x + yi + zj + wk \mapsto x - yi - zj - wk$. Define the norm $N : B \to \mathbb{Q}$ to be $\mathrm{N}(b) = bb^*$.

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- Endow B with the Galois action inherited from E, concretely $\sigma(x + yi + zj + wk) = \sigma(x) + \sigma(y)i + \sigma(z)j + \sigma(w)k$.
- Define

$$X := \{b \in B \mid b^* = \sigma(b)\}$$
$$= \{x + y\sqrt{d}i + z\sqrt{d}j + w\sqrt{d}k \mid x, y, z, w \in \mathbb{Q}\}$$

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An exact sequence

- Let $\psi : E^\times \to \mathbb{Q}^\times \times B^\times$ be defined by $r \mapsto (\mathrm{N}_{\mathbb{Q}}^{E}(r), r)$ and let $\rho : \mathbb{Q}^\times \times B^\times \to \mathrm{GSO}(X)$ be defined by $\rho(t,b) \cdot x = t^{-1} b x \sigma(b)^*$. Knus showed that the following sequence is exact:

$$1 \to E^\times \xrightarrow{\psi} \mathbb{Q}^\times \times B^\times \xrightarrow{\rho} \mathrm{GSO}(X) \to 1$$

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Let $\mathcal{S}(X_v^2)$ be the space of locally constant functions with compact suppoert $X_v^2 \to \mathbb{C}$ when v is a finite place, and to be the set of functions which rapidly decay away from 0 when v is infinite. We have the formulas for the Weil representation ω of $\mathrm{Sp}(4, \mathbb{Q}_v) \times \mathrm{O}(X_v)$ on $\mathcal{S}(X_v^2)$ which can be extended to $R \subset \mathrm{GSp}(4, \mathbb{Q}_v) \times \mathrm{GO}(X_v)$ where the similitude factors match.

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$$(\omega(1,h)\varphi)(x_1,x_2) = \varphi(h^{-1}x_1, h^{-1}x_2),$$

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$$= \chi_X(a_1a_4 - a_2a_3)|a_1a_4 - a_2a_3|^{\dim X/2}\varphi(a_1x_1 + a_3x_2, a_2x_1 + a_4x_2),$$

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Theta Series

- Now fix $\varphi \in \mathcal{S}(X_v^2)$ for each place and set $\varphi = \otimes_v \varphi_v$ where each $\varphi_v \in \mathcal{S}(X_v^2)$.
- Then for $(g, h) \in R \subset \mathrm{GSp}(4, \mathbb{A}) \times \mathrm{GO}(X(\mathbb{A}))$, we have the map $\omega(g, h)(\varphi) : X(\mathbb{A})^2 \to \mathbb{C}$

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- Define the following series, which is absolutely convergent and is left $R(\mathbb{Q})$ invariant:

$$\vartheta(g, h; \varphi) := \sum_{x \in X(\mathbb{Q})^2} (\omega(g, h)\varphi)(x_1, x_2)$$

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$$\vartheta(g, h; \varphi) := \sum_{x \in X(\mathbb{Q})^2} \big(\omega(g, h)\varphi\big)(x_1, x_2)$$

Theta Series

- Now fix $\varphi \in \mathcal{S}(X_v^2)$ for each place and set $\varphi = \otimes_v \varphi_v$ where each $\varphi_v \in \mathcal{S}(X_v^2)$.
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Theta Lift

Let f be a cusp form on $\mathrm{GO}(X,\mathbb{A})$ of trivial central character and $\varphi \in \mathcal{S}(X(\mathbb{A})^2)$. Let $\mathrm{GSp}(4,\mathbb{A})^+$ be the subgroup of $g \in \mathrm{GSp}(4,\mathbb{A})$ such that $\lambda(G) \in \lambda(\mathrm{GO}(X,\mathbb{A}))$. For $g \in \mathrm{GSp}(4,\mathbb{A})^+$ define:

$$\theta(f,\varphi)(g) = \int_{\mathrm{O}(X,\mathbb{Q})\backslash \mathrm{O}(X,\mathbb{A})} \vartheta(g, h_1 h; \varphi) f(h_1 h) dh_1$$

where $h \in \mathrm{GO}(X,\mathbb{A})$ is any element such that $(g,h) \in R(\mathbb{A})$. Then θ can be extended uniquely to all of $\mathrm{GSp}(4,\mathbb{A})$ which is left invariant under $\mathrm{GSp}(4,\mathbb{Q})$ and is, in fact, an automorphic form on $\mathrm{GSp}(4,\mathbb{A})$.

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Putting the maps together

- $\iota : B^\times \backslash B(\mathbb{A})^\times / R(\mathbb{A})^\times \xrightarrow{\sim} R^\times \backslash P^1(\mathfrak{O}/\mathfrak{q})$
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Example: $\mathbb{Q}(\sqrt{5})$

- Set $E = \mathbb{Q}(\sqrt{5})$ and $D = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, the classical set of hamiltonians. It is well know that $D_v := D \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is a division algebra exactly when $v = 2$ or $v = \infty$. While $B := D \otimes_{\mathbb{Q}} E$ has discriminate equal to (1) because it is division precisely at the two infinite places.
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Model for X when $5 \in \mathbb{Q}_v^2$

In this case we have that $X \cong \{(d, d^*) \mid d \in D\} = D$ which yields the following commutative diagram.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathbb{Q}_v^\times \times \mathbb{Q}_v^\times & \longrightarrow & \mathbb{Q}_v^\times \times B_v^\times & \xrightarrow{\rho} & \mathrm{GSO}(X_v) & \longrightarrow & 1 \\
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- For all such primes we give an isomorphism $\varphi : D_v \to M_2(\mathbb{Q}_v)$ which extends to a map $\varphi : B_v \to M_2(\mathbb{Q}_v(\sqrt{5}))$:

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Where $-1 = x^2 + y^2$. So then the following diagrams commute:

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- The next challenge is choosing the Schwarz functions. The theorem of Johnson-Leung and Roberts guarantees that we can make choices so that the resulting theta lift is actually a paramodular form.
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Paramodular Group

- Recall the symplectic group is defined to be $\mathrm{Sp}(4) := \{g \in \mathrm{GL}(4) \mid {}^t gJg = J\}$ where $J = \begin{bmatrix} 0 & I_2 \\ -I_2 & 0 \end{bmatrix}$
- The *paramodular group* is the subgroup of $\mathrm{Sp}(4)$ given by

$$\Gamma_{para}(N) := \mathrm{Sp}(4) \cap \begin{bmatrix} \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} \\ \mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} \end{bmatrix}$$

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Conjecture (Brumer & Kramer, 2010)

There is a one-to-one correspondence between isogeny classes of abelian surfaces over $\mathbb{Q}$ *with conductor* N *and not of* GL_2 *type with Paramodular newforms of level* N *with rational eigenvalues, up to scalar multiples, which are not Gritsenko lifts.*

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